The existence of traveling wave solutions for a three species competition system

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Outline

Introduction

- In a 1D habitat, there are three species u, v and w living together such that each species has the preference of food resource so that the competition occurs only between species u and v and between species v and w, respectively.
- In other words, species u and w have different preferences of food resource so that no competition between them.
- But, species v has both preferences so that it needs to compete with both species u and w.

So we have the following three species competition system:

$$u_t = D_1 u_{xx} + r_1 u(1 - u - b_{12} v), \quad x, t \in \mathbf{R}, \tag{1}$$

$$v_t = D_2 v_{xx} + r_2 v(1 - b_{21} u - v - b_{23} w), \quad x, t \in \mathbf{R}, \tag{2}$$

$$w_t = D_3 w_{xx} + r_3 w(1 - b_{32} v - w), \quad x, t \in \mathbf{R}, \tag{3}$$

where $D_i > 0, r_i > 0, b_{ij} > 0$.

- Here u, v, w are the population densities of species $1, 2, 3$,
- b_{ij} is the competition coefficient of species j to species i,
- r_i is the net growth rate of species i,
- D_i is the diffusion coefficient of species i.
- Also, we have taken the scales of species so that the carrying capacity of each species is normalized to be 1,
- the states $(u, v, w) = (0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 0, 1)$ are equilibria of the system (1)-(3).

We shall always assume that

(A) $b_{12}, b_{32} > 1,\ b_{21} + b_{23} < 1,$

which means that the species u, w are weak competitors to the species v.

Intuitively, species v will win the competition and wipe out both species u and w eventually.

Therefore, the aim of this work is to study the existence of traveling wave for (continuous) model (1)-(3) in the form

$$(u, v, w)(x, t) := (\varphi, \psi, \theta)(y), \quad y := x + st,$$

where s is the wave speed and (φ, ψ, θ) is the wave profile.

We are interested in the monotone wave connecting the equilibria $(1, 0, 1)$ and $(0, 1, 0)$.
Hence it is reduced to the study of the following problem:

$$\begin{cases} s\varphi' = D_1\varphi'' + r_1\varphi(1 - \varphi - b_{12}\psi), \quad y \in \mathbf{R}, \\ s\psi' = D_2\psi'' + r_2\psi(1 - b_{21}\varphi - \psi - b_{23}\theta), \quad y \in \mathbf{R}, \\ s\theta' = D_3\theta'' + r_3\theta(1 - b_{32}\psi - \theta), \quad y \in \mathbf{R}, \\ (\varphi, \psi, \theta)(-\infty) = (1, 0, 1), \quad (\varphi, \psi, \theta)(+\infty) = (0, 1, 0), \\ 0 \le \varphi, \psi, \theta \le 1 \end{cases} \tag{4}$$

for certain speed $s > 0$.

- The problem (4) is a 6-dim dynamical system.

Notice that, by a linearization of the corresponding kinetic systems to (1)-(3), we can easily check that near the equilibrium $(1, 0, 1)$ the stable manifold is of dimension 2 and the unstable manifold is of dimension 1; and the equilibrium $(0, 1, 0)$ is stable such that the stable manifold is of dimension 3, under the assumption **(A).**

If we consider the linearization of the second equation of (4) around the state $(1, 0, 1)$, the corresponding characteristic equation is given by

$$D_2\mu^2 - s\mu + r_2(1 - b_{21} - b_{23}) = 0. \tag{5}$$

We easily obtain that (5) has a positive solution if and only if $s \geq s_*$, where

$$s_* := 2\sqrt{D_2 r_2(1 - b_{21} - b_{23})}.$$

Thus, the minimal speed $s_{\min}$ (if it exists) for the continuous model (1)-(3) with $\varphi' < 0, \psi' > 0, \theta' < 0$, must satisfy that $s_{\min} \geq s_*$.

Indeed, since the limiting linear equation of the second equation in (4) as $y \to -\infty$ is given by

$$D_2\psi'' - s\psi' + r_2(1 - b_{21} - b_{23})\psi = 0$$

which has a monotone solution near $y = -\infty$ only if $s \geq s_*$. Hence we should have $s_{\min} \geq s_*$.

We remark that the minimal speed $s_{\min}$ is the constant such that a traveling wave solution with speed s exists if and only if $s \geq s_{\min}$.

We now state the following main result of this work on the linear determinacy for (4).

Theorem 1

Assume that **(A)** *holds. Also, let* $D_2, r_2, b_{21}, b_{23} > 0$ *be given. Then* $s_{\min} = s_*$ *as long as* $(D_j, r_j, b_{j2}) \in B_j^1 \cup B_j^2$, $j = 1, 3$, *where*

$$B_j^1 := \{D_j \in (0, 2D_2],\ b_{j2}(b_{21} + b_{23}) \leq 1,\ r_j > 0\}, \tag{6}$$

$$B_j^2 := \Big\{D_j \in (0, 2D_2),\ b_{j2}(b_{21} + b_{23}) > 1,\ 0 < r_j < \left(2 - \frac{D_j}{D_2}\right) \frac{r_2(1 - b_{21} - b_{23})}{b_{j2}(b_{21} + b_{23}) - 1}\Big\}, \tag{7}$$

for $j = 1, 3$.

- In fact, the definition of linear determinacy is first defined in [Lewis-Li-Weinberger(2002)], which means that the minimal speed is determined by the linearization of the problem at some unstable equilibrium.
- For the works related to linear determinacy, we refer to [Hosono(1998)], [Huang(2010)], [Huang-Han(2011)], [Guo-Liang(2011)] for a 2-species competition system.

Main idea for the proof of Theorem 1

First, we derive the linear determinacy of the spatial discretization of (1)-(3) in the following form

$$u_j'(t) = d_1\mathcal{D}[u_j](t) + r_1 u_j(t)[1 - u_j(t) - b_{12}v_j(t)], \tag{8}$$
$$v_j'(t) = d_2\mathcal{D}[v_j](t) + r_2 v_j(t)[1 - b_{21}u_j - v_j(t) - b_{23}w_j(t)], \tag{9}$$
$$w_j'(t) = d_3\mathcal{D}[w_j](t) + r_3 w_j(t)[1 - b_{32}v_j(t) - w_j], \tag{10}$$

for $j \in \mathbf{Z}$, $t \in \mathbf{R}$, where d_j is the discrete diffusion rate and $\mathcal{D}[u_j] := (u_{j+1} - u_j) + (u_{j-1} - u_j)$ ...

- In particular, we take $d_j = D_j/\tau^2$ in (8)-(10) for any $\tau > 0$ small.
- Hence we have a sequence of traveling waves for the approximated discrete problems.
- In order to passing to the limit, we apply the method of discrete Fourier transform to derive the equi-continuity of the approximation sequence of wave profiles.
- Such an indirect approach (used first in [G.-Liang(2011)]) might be unusual, but it has its own interest and advantage.

The details

Now, a traveling wave of (8)-(10) is a solution in the form

$$(u_j(t), v_j(t), w_j(t)) = (U(\xi), V(\xi), W(\xi)), \quad \xi = j + ct,$$

where c is the wave speed and $\{U, V, W\}$ are the wave profiles.

Therefore, the problem of finding traveling wave of (8)-(10) is equivalent to find $(c, U, V, W) \in \mathbf{R} \times [C^1(\mathbf{R})]^3$ such that

$$\begin{cases} cU' = d_1\mathcal{D}[U] + r_1U(1 - U - b_{12}V), \quad \xi \in \mathbf{R}, \\ cV' = d_2\mathcal{D}[V] + r_2V(1 - b_{21}U - V - b_{23}W), \quad \xi \in \mathbf{R}, \\ cW' = d_3\mathcal{D}[W] + r_3W(1 - b_{32}V - W), \quad \xi \in \mathbf{R}, \\ (U, V, W)(-\infty) = (1, 0, 1), \quad (U, V, W)(+\infty) = (0, 1, 0), \\ 0 \le U, V, W \le 1, \end{cases} \tag{11}$$

where $\mathcal{D}[u](\xi) := u(\xi + 1) + u(\xi - 1) - 2u(\xi)$ etc.

The first result shows the existence of the minimal wave speed for (11).

Theorem 2

Assume **(A)**. *Then there exists a positive constant* $c_{\min}$ *such that the problem* (11) *admits a solution* (c, U, V, W) *satisfying* $U'(\cdot) < 0$, $V'(\cdot) > 0$ *and* $W'(\cdot) < 0$ *on* $\mathbb{R}$ *if and only if* $c \geq c_{\min}$.

- The related works about the minimal speed for lattice dynamical systems can be found in, for example, [Chen-G.(2002)], [Chen-G.(2003)], [G.-Hamel(2006)], [G.-Wu(2008)], [G.-Wu(2012)].

Proof of Theorem 2

- We transform the problem into a monotone system.
- If we can find a suitable pair of super-sub-solutions, then we can apply the classical monotone iteration scheme.
- An idea developed in [Chen-G.(2003)], we study a sequence of truncated problems in which only a super-solution is needed.
- This method (of truncation) was applied to the two component LDS in [G.-Wu(2012)].

- The main difficulty here is to make sure the limit satisfies the desired boundary conditions at $\pm\infty$..
- To overcome this difficulty, we introduce one condition in the definition of super-solution as

$$U_+(\xi_0) < 1,\ W_+(\xi_0) < 1 \quad \text{for some } \xi_0 \in \mathbf{R},$$

instead of non-constant-ness in the previous works.

To estimate the minimal speed of (11), we define

$$c_* := \inf_{\lambda>0}\left\{\frac{d_2(e^{\lambda}+e^{-\lambda}-2)+r_2(1-b_{21}-b_{23})}{\lambda}\right\}.$$

It is clear that

$$c\lambda = d_2(e^{\lambda}+e^{-\lambda}-2)+r_2(1-b_{21}-b_{23}) \tag{12}$$

has a positive solution if and only if $c \geq c_*$.
Moreover, there exists $\lambda_* > 0$ such that λ_* is the unique solution of (12) when $c = c_*$. For $c > c_*$, (12) has exactly two solutions $\lambda_i(c)$, $i = 1, 2$, with $0 < \lambda_1(c) < \lambda_2(c)$.

Theorem 3 (Chen-G. (2003), Chen-Fu-G. (2006))

Let $c > 0$ be a constant and $B(\cdot)$ be a continuous function having finite $B(\pm\infty) := \lim_{x\to\pm\infty} B(x)$. Let $z(\cdot)$ be a measurable function satisfying

$$c\, z(x) = e^{\int_x^{x+1} z(s)ds} + e^{-\int_{x-1}^{x} z(s)ds} + B(x) \quad \forall\, x \in \mathbf{R}.$$

Then z is uniformly continuous and bounded.
In addition, $\omega^{\pm} = \lim_{x\to\pm\infty} z(x)$ exist and are roots of the characteristic equation

$$c\,\omega = e^{\omega} + e^{-\omega} + B(\pm\infty).$$

Apply this fundamental theorem to

$$cV' = d_2\mathcal{D}[V] + r_2V(1 - b_{21}U - V - b_{23}W)$$

with the ratio $z := V'/V$ at $\xi = -\infty$, we have

Theorem 4

Assume **(A)***. Then* $c_{\min} \geq c_*$.

By applying an idea used in [G.-Liang(2011)] ([G.-Wu(2012)]), the linear determinacy for (11) is given as follows:

Theorem 5

Assume **(A)**. *Let* $r_2 > 0$, $b_{21} > 0$ *and* $b_{23} > 0$ *be given. Then there exists a constant* $d_* = d_*(d_2) > 2d_2$ *such that* $c_{\min} = c_*$ *as long as* $(d_j, r_j, b_{j2}) \in A_j^1 \cup A_j^2$, $j = 1, 3$, *where*

$$A_j^1 := \{d_j \in (0, d_*],\ b_{j2}(b_{21} + b_{23}) \leq 1,\ r_j > 0\}, \qquad (13)$$

$$A_j^2 := \Big\{d_j \in (0, d_*],\ b_{j2}(b_{21} + b_{23}) > 1, \quad 0 < r_j \leq \frac{d_* - d_j}{d_* - d_2} \cdot \frac{r_2(1 - b_{21} - b_{23})}{b_{j2}(b_{21} + b_{23}) - 1}\Big\} \qquad (14)$$

for $j = 1, 3$.

A detailed analysis of the quantity $d_*(d_2)$, we have

Lemma 6

Suppose that $(D_j, r_j, b_{j2}) \in B_j^1 \cup B_j^2$ *for* $j = 1, 3$. *Let* $d_j(\tau) := D_j/\tau^2$ *and* $d_*(\tau) := d_*(d_2(\tau))$ *for* $\tau > 0$. *Then* $(d_j(\tau), r_j, b_{j2}) \in A_j^1 \cup A_j^2$ *for* $j = 1, 3$, *for all small* $\tau > 0$.

- Then a suitable approximated sequence of wave profiles for the discrete problems can be chosen and Theorem 1 can be proved.

Discussion

- Due to the structure of the nonlinear terms, the corresponding dynamical system is monotone so that our method can be applied.
- Indeed, we can prove the existence of traveling waves for the continuous system (1)-(3) with more general parameters.
- For the special case (as in Theorem 1), we are able to derive the existence and non-existence of traveling waves. In other words, the minimal speed is given exactly.

- In general, there is no comparison for the 3 species competition system (so that the corresponding system is not monotone and our method cannot be applied).
- For certain general non-monotone system, we refer to the recent works by C.-C. Chen and his co-authors.